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Möbius function and characteristic monomials for combinatorial enumeration

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Abstract. After the definitions of amplified representations and number-theoretical vectors, the markaracter table of a cyclic subgroup is converted into the corresponding **Q**-conjugacy character table. The conversion is shown to necessitate an interconversion matrix that contains Möbius functions as elements. Since the interconversion matrix gives characteristic monomials for cyclic groups, all the powers appearing in each of the characteristic monomials are shown to be integers. Characteristic monomials for finite groups are then built up by starting from those of cyclic groups. This procedure clarifies the fact that all the powers appearing in each characteristic monomial for finite groups are integers. The relationship between characteristic monomial tables and unit-subduced-cycle-index tables is discussed with respect to their application to isomer enumeration.

Key words: **Q**-conjugacy character – Characteristic monomial – Möbius function – Enumeration – Group

1 Introduction

1.1 Background

Chemical applications of group theory can be categorized into two distinct approaches. One approach has been the application to quantum chemistry, spectroscopy, etc., where linear representations, irreducible representations, and character tables play an important role, as explained in many excellent textbooks [1–9]. The other approach has mainly aimed at the enumeration of isomers [10–14] though the methods concerned have been further applied to enumeration problems in quantum chemistry, etc., as reviewed in detail in Ref. [15]. In this approach, the concept of cycle indices is a key to counting isomers with respect to molecular formulas. The approach has been extended to make it capable of more elaborate enumerations concerning both molecular formulas and symmetries [16–19], which are based on permutation representations and mark tables [20–23]. We have recently reported the unit-subduced-cycle-index (USCI) approach to systematic enumeration of isomers, where subduction of coset representations is proposed as a new methodology [24–30].

1.2 Problem setting

In the previous papers of this series [29, 30], we proposed the concept of markaracters, which was later shown to link characters with marks via **Q**-conjugacy characters [31]. The latter characters have been defined as matured characters related to **Q**-conjugacy classes. Thus, the two approaches described above can now be discussed on a common basis. In particular, we have shown that any character table (e.g., Table 1 for **T** group) can be transformed to the corresponding **Q**-conjugacy character table (e.g., Table 2 for **T** group) by considering **Q**-conjugacy classes [e.g., $\mathbf{K}_1$, $\mathbf{K}_2$, and $\mathbf{K}_3(=\mathbf{K}_{31}+\mathbf{K}_{31})$] in place of conjugacy classes (e.g., $\mathbf{K}_1=\{I\}$, $\mathbf{K}_2=\{C_{2(1)},C_{2(2)},C_{2(3)}\}$, $\mathbf{K}_{31}=\{C_{3(1)},C_{3(2)},C_{3(3)},C_{3(4)}\}$ and $\mathbf{K}_{32}=\{C^2_{3(1)},C^2_{3(2)},C^2_{3(3)},C^2_{3(4)}\}$) [31, 32]. Moreover, characteristic monomial tables (e.g., Table 3 for **T** group) have been derived from **Q**-conjugacy character tables, where they have been used for solving enumeration problems in place of USCIs derived from markaracters.

The remaining problem is to clarify the properties of the characteristic monomials. As found by comparison between Tables 2 and 3, the power of each dummy variable of subscript 1 (s_1) is equal to the corresponding **Q**-conjugacy character. However, it has not been clarified whether the power of a dummy variable of subscript 2 or more (s_n for $n \geq 2$) is an integer or not. In this paper, we shall prove that all the powers appearing in each characteristic monomial are integers and shall discuss the relationship between characteristic monomial tables and USCI tables. This proof assures us of the wide applicability of characteristic monomials to various problems of combinatorial enumeration.

Table 1. Character table for **T**[a]

	$\mathbf{K}_1$	$\mathbf{K}_2$	$\mathbf{K}_{31}$	$\mathbf{K}_{32}$
A	1	1	1	1
$E_{(a)}$	1	1	ω	ω^2
$E_{(b)}$	1	1	ω^2	ω
T	3	−1	0	0

[a] $\omega = \cos(2\pi/3) + i\sin(2\pi/3)$

Table 2. **Q**-conjugacy character table for **T**

	$\mathbf{K}_1$ $\mathbf{C}_1$	$\mathbf{K}_2$ $\mathbf{C}_2$	$\mathbf{K}_3$ $\mathbf{C}_3$
A	1	1	1
$E(=\mathrm{E}_{(a)}+E_{(b)})$	2	2	−1
T	3	−1	0

Table 3. Characteristic monomial table for **T**

	$\downarrow\mathbf{C}_1$	$\downarrow\mathbf{C}_2$	$\downarrow\mathbf{C}_3$
A	s_1	s_1	s_1
E	s_1^2	s_1^2	$s_1^{-1}s_3$
T	s_1^3	$s_1^{-1}s_2^2$	s_3
N_{j}	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{2}{3}$

2 Number-theoretic vectors

The purpose of this section is to describe the concept of number-theoretic vectors in order to clarify the relationship between markaracters [29] and **Q**-conjugacy characters [31]. The concept is an extention of number-theoretic functions described in Chapter 1 of Ref. [33]. Although the main result described in this section has been well known [34], an alternative viewpoint concerning markaracter tables and **Q**-conjugacy characters tables will be mentioned for the application to combinatorial enumeration.

2.1 Amplificative equivalence of coset representations

Let $\mathbf{C}_n$ be a cyclic group of order n. Any subgroup of $\mathbf{C}_n$ is a cyclic group of order d, where the integer d is a divisor of n. Then, we have a coset decomposition,

$$\mathbf{C}_n = \mathbf{C}_d h_1 + \mathbf{C}_d h_2 + \cdots + \mathbf{C}_d h_{s/d}, \tag{1}$$

where $h_1, h_2, \ldots, h_{s/d}$ are the representatives of the cosets. The coset decomposition produces a coset representation $\mathbf{C}_n(/\mathbf{C}_d)$. Obviously, $\mathbf{C}_n(/\mathbf{C}_d)$ is not faithful if d is not equal to 1. For any $h \in \mathbf{C}_d$ and any coset $\mathbf{C}_d h_i$ $(i = 1, 2, \ldots, s/d)$, we have

$$\mathbf{C}_d h_i h = \mathbf{C}_d h h_i = \mathbf{C}_d h_i. \tag{2}$$

It follows that $\mathbf{C}_n(/\mathbf{C}_d)(h) = I$ for any $h \in \mathbf{C}_d$, where I is an identity permutation of degree s/d. On the other hand, we have

$$\mathbf{C}_d h_i h = \mathbf{C}_d h h_i \neq \mathbf{C}_d h_i \tag{3}$$

for any $h \notin \mathbf{C}_d$. As a result, we obtain a lemma.

Lemma 1. *Let $\mathbf{C}_d$ be a cyclic subgroup of a cyclic group of $\mathbf{C}_n$. Then, the kernel of the corresponding coset representation $\mathbf{C}_n(/\mathbf{C}_d)$ is the cyclic subgroup $\mathbf{C}_d$ itself.*

Since any $\mathbf{C}_d$ is a normal subgroup of $\mathbf{C}_n$, the coset decomposition represented by Eq. (1) produces the corresponding factor group:

$$\mathbf{C}_n/\mathbf{C}_d = \{\mathbf{C}_d h_1, \mathbf{C}_d h_2, \ldots, \mathbf{C}_d h_{s/d}\}. \tag{4}$$

Lemma 1 allows us to equate the coset representation $\mathbf{C}_n(/\mathbf{C}_d)$ with the factor group $\mathbf{C}_n/\mathbf{C}_d$ and with neglect of the unfaithfullness of $\mathbf{C}_n(/\mathbf{C}_d)$.

Lemma 2. *Let $\mathbf{C}_d$ be a cyclic subgroup of a cyclic group of $\mathbf{C}_n$. Then, the corresponding factor group $\mathbf{C}_n/\mathbf{C}_d$ is isomorphic to a cyclic group of order n/d, i.e.,*

$$\mathbf{C}_n/\mathbf{C}_d \cong \mathbf{C}_{n/d}. \tag{5}$$

Proof. Let $\tilde{h}$ be a generator of $\mathbf{C}_n$, i.e., $\langle\tilde{h}\rangle = \mathbf{C}_n$. Then, we select a coset $\mathbf{C}_d\tilde{h}$, which satisfies

$$(\mathbf{C}_d\tilde{h})^r = \mathbf{C}_d\tilde{h}^r \tag{6}$$

for $r = 1, 2, \ldots, n$. Since $\tilde{h}$ is a generator of $\mathbf{C}_n$, Eq. (6) covers all of the cosets represented by Eq. (1) when the integer r runs from 1 to n. Because $n/d \leq n$, the set of cosets represented by Eq. (6) has a redundancy. Since the generator of $\mathbf{C}_d$ is $\tilde{h}^{n/d}$, we have

$$\mathbf{C}_d = \{\tilde{h}^{n/d}, \tilde{h}^{2n/d}, \tilde{h}^{3n/d}, \ldots, \tilde{h}^{d\times n/d}\}. \tag{7}$$

Hence, the redundancy in Eq. (6) $(r = 1, 2, \ldots, n)$ can be eliminated as follows.

Let us consider a set of cosets from $\mathbf{C}_d\tilde{h}^{an/d}$ to $\mathbf{C}_d\tilde{h}^{(an+1)/d}$. Then, we have

$$\mathbf{C}_d\tilde{h}^{an/d+b} = \mathbf{C}_d\tilde{h}^{an/d}\tilde{h}^b = \mathbf{C}_d\tilde{h}^b, \tag{8}$$

for $b = 1, 2, \ldots, n/d$. As a runs from 0 to $d-1$, the cosets of Eq. (6) remain identical with $\mathbf{C}_d\tilde{h}^b$ independent of a. As a result, the factor group $\mathbf{C}_n/\mathbf{C}_d$ without redundancy is obtained as follows.

$$\mathbf{C}_n/\mathbf{C}_d = \{\mathbf{C}_d\tilde{h}^1, \mathbf{C}_d\tilde{h}^2, \ldots, \mathbf{C}_d\tilde{h}^{n/d}\}, \tag{9}$$

where $\tilde{h}^{n/d} \in \mathbf{C}_d$. Because of Eq. (6), Eq. (9) shows that the factor group $\mathbf{C}_n/\mathbf{C}_d$ is a cyclic group of order n/d.

Example 1. All of the coset representations of the point group $\mathbf{C}_6$ (a cyclic group of order 6) are collected in Table 4, in which their kernels are shown as examples of Lemmas 1 and 2. Each element of a kernel has a permutation expressed as a set of 1-cycles, e.g., (1)(2)(3)

Table 4. Coset representations for $\mathbf{C}_6$

Element	$\mathbf{C}_6(/\mathbf{C}_1)$	$\mathbf{C}_6(/\mathbf{C}_2)$	$\mathbf{C}_6(/\mathbf{C}_3)$	$\mathbf{C}_6(/\mathbf{C}_6)$
$\mathbf{C}_6$	(1 2 3 4 5 6)	(1 3 2)	(1 2)	(1)
$\mathbf{C}_3$	(1 3 5)(2 4 6)	(1 2 3)	(1)(2)	(1)
$\mathbf{C}_2$	(1 4)(2 5)(3 6)	(1)(2)(3)	(1 2)	(1)
$\mathbf{C}_3^2$	(1 5 3)(2 6 4)	(1 3 2)	(1)(2)	(1)
$\mathbf{C}_6^5$	(1 6 5 4 3 2)	(1 2 3)	(1 2)	(1)
1	(1)(2)(3)(4)(5)(6)	(1)(2)(3)	(1)(2)	(1)
Kernel	$\mathbf{C}_1$	$\mathbf{C}_2$	$\mathbf{C}_3$	$\mathbf{C}_6$

appearing in the $\mathbf{C}_6(/\mathbf{C}_2)$ column for the I or C_2 element. As a result, $\mathbf{C}_2 = \{I, C_2\}$ is the kernel of $\mathbf{C}_6(/\mathbf{C}_2)$. When we carry out the coset decomposition of the group $\mathbf{C}_6$ by the kernel $\mathbf{C}_2$, we have three cosets, where the coset $\mathbf{C}_2 I = \{I, C_2\}$ corresponds to the cycle (1)(3)(2), $\mathbf{C}_2 C_3 = \{C_3, C_6^5\}$ to (1 2 3) and $\mathbf{C}_2 C_3^2 = \{C_3^2, C_6\}$ to (1 3 2), as found in Table 4.

Let us consider the regular representation of $\mathbf{C}_{n/d}$, i.e., $\mathbf{C}_{n/d}(/\mathbf{C}_1)$. This is related to the factor group $\mathbf{C}_n/\mathbf{C}_d$ (Lemma 2), which is in turn related to the coset representation $\mathbf{C}_n(/\mathbf{C}_d)$ (Lemma 1). We here use the symbol $\mathbf{C}_n(/\mathbf{C}_d)(h)$ to represent the permutation of the coset representation $\mathbf{C}_n(/\mathbf{C}_d)$ for an element h ($\in \mathbf{C}_n$). For example, we have $\mathbf{C}_6(/\mathbf{C}_2)(I) = (1)(2)(3)$ and $\mathbf{C}_6(/\mathbf{C}_2)(C_2) = (1)(2)(3)$, as found in Table 4. Thus, Lemmas 1 and 2 give a theorem.

Theorem 1 (amplified coset representations). *Let $\mathbf{C}_n$ be a cyclic group of order n. Then, a coset representation $\mathbf{C}_n(/\mathbf{C}_d)$ for the group $\mathbf{C}_n$ is obtained from the regular representation $\mathbf{C}_{n/d}(/\mathbf{C}_1)$ of the subgroup $\mathbf{C}_{n/d}$ by placing*

$$\mathbf{C}_n(/\mathbf{C}_d)(h) = \mathbf{C}_{n/d}(/\mathbf{C}_1)(\tilde{h}^b) \quad \text{for} \quad h \in \mathbf{C}_d\tilde{h}^b, \tag{10}$$

where the element $\tilde{h}$ is the generator of the cyclic group $\mathbf{C}_{n/d}$ and the integer b runs over the range $b = 1, 2, \ldots, n/d$. The resulting representation is called an amplified coset representation.

Theorem 1 enables us to equate $\mathbf{C}_n(/\mathbf{C}_d)$ with $\mathbf{C}_{n/d}(/\mathbf{C}_1)$ in terms of the amplification procedure. This situation is symbolically denoted by the following equation.

$$\mathbf{C}_n(/\mathbf{C}_d) \overset{\text{amp}}{=} \mathbf{C}_{n/d}(/\mathbf{C}_1). \tag{11}$$

2.2 Amplificative equivalence of irreducible representations

Let $\tilde{h} = C_n$ be a generator of a cyclic group $\mathbf{C}_n$, which has irreducible representations

$$\Gamma_{C_n^d} = \{\varepsilon^d, (\varepsilon^d)^2, \ldots, (\varepsilon^d)^n\} \tag{12}$$

for $d = 1, 2, \ldots, n$, where we have $\varepsilon = \cos(2\pi/n) + i\sin(2\pi/n)$. The element $(\varepsilon^d)^r$ corresponds to the element $\tilde{h}^r = (C_n^d)^r$, i.e.,

$$\Gamma_{C_n^d}(\tilde{h}^r) = \varepsilon^{rd}. \tag{13}$$

Since Eq. (13) gives $\Gamma_{C_n^d}(\tilde{h}^{n/d}) = \varepsilon^n = 1$ for $r = n/d$, we have

$$\Gamma_{C_n^d}(\tilde{h}^{an/d}) = \Gamma_{C_n^d}(\tilde{h}^{n/d})^a = (\varepsilon^n)^a = 1, \tag{14}$$

where a runs from 1 to d. Let us consider $\tilde{h}^r$ is equal to an element selected from $\tilde{h}^{an/d+b}$ for $b = 1, 2, \ldots, n/d$. By virtue of Eq. (14), Eq. (13) is transformed into

$$\begin{aligned}\Gamma_{C_n^d}(\tilde{h}^{an/d+b}) &= \Gamma_{C_n^d}(\tilde{h}^{an/d})\Gamma_{C_n^d}(\tilde{h}^b) \\ &= \Gamma_{C_n^d}(\tilde{h}^b) = (\varepsilon^d)^b,\end{aligned} \tag{15}$$

for $b = 1, 2, \ldots, n/d$. Since ε is an nth primitive root of 1, we have $(\varepsilon^d)^{n/d} = \varepsilon^n = 1$, which means that ε^d is an n/dth primitive root of 1. Hence, the last side of Eq. (15) gives a representation of $\mathbf{C}_{n/d}$ when b runs from 1 to n/d, i.e.,

$$\Gamma_{C_{n/d}} = \{\varepsilon^d, (\varepsilon^d)^2, \ldots, (\varepsilon^d)^{n/d}\}, \tag{16}$$

or equivalently

$$\Gamma_{C_{n/d}}(\tilde{h}^b) = (\varepsilon^d)^b, \tag{17}$$

for $b = 1, 2, \ldots, n/d$. Comparison of Eqs. (15) and (17) reveals that $\Gamma_{C_n^d}$ for the group $\mathbf{C}_n$ is obtained from $\Gamma_{C_{n/d}}$ for the group $\mathbf{C}_{n/d}$ by an amplification procedure in which the integer a runs from 0 to $d-1$. By analogy with Theorem 1, we have a theorem concerning amplified irreducible representations.

Theorem 2 (amplified irreducible representations). *Let $\mathbf{C}_n$ be a cyclic group of order n. Then, an irreducible representation $\Gamma_{C_{n/d}}$ for the group $\mathbf{C}_{n/d}$ is amplified into $\Gamma_{C_n^d}$ for the group $\mathbf{C}_n$ by placing*

$$\Gamma_{C_n^d}(\tilde{h}^{an/d+b}) = \Gamma_{C_{n/d}}(\tilde{h}^b) = (\varepsilon^d)^b \tag{18}$$

where the integer a runs over the range $a = 0, 1, 2, \ldots, d-1$, and the integer b runs over the range $b = 1, 2, \ldots, n/d$. The resulting representation is called an amplified irreducible representation.

By collecting the elements of Eq. (15) satisfying $b = n/d$, we have the subgroup $\mathbf{C}_d$ (Eq. 7). Since we have $(\varepsilon^d)^{n/d} = \varepsilon^n = 1$, Eq. (18) shows that all of the elements contained in $\mathbf{C}_d$ have an irreducible representation of unity. This fact corresponds to lemma 1. Hence, Eq. (18) of Theorem 2 can be transformed as follows by using the notation used in Theorem 1:

$$\Gamma_{C_n^d}(h) = \Gamma_{C_{n/d}}(\tilde{h}^b) \quad \text{for} \quad h \in \mathbf{C}_d\tilde{h}^b, \tag{19}$$

where the integer b runs over the range $b = 1, 2, \ldots, n/d$.

Theorem 2 enables us to equate $\Gamma_{C_n^d}$ for the group $\mathbf{C}_n$ with $\Gamma_{C_{n/d}}$ for the group $\mathbf{C}_{n/d}$ in terms of the amplification procedure. This situation is symbolically denoted by the following equation.

$$\Gamma_{C_n^d} \overset{\text{amp}}{=} \Gamma_{C_{n/d}}. \tag{20}$$

Example 2. Let us consider an irreducible representation, $\Gamma_{C_6^2}$, for a cyclic group $\mathbf{C}_6$, where the value ε^2 is assigned to a generator C_6. Note that ε is the sixth primitive root of 1. Then, $\Gamma_{C_6^2}$ can be schematically represented by the left half of the following illustration:

$$\begin{array}{cccccccccc} & C_6 & C_6^2 & C_6^3 & & & & & & \\ & \updownarrow & \updownarrow & \updownarrow & & & & & & \\ \Gamma_{C_6^2} & \varepsilon^2 & (\varepsilon^2)^2 & (\varepsilon^2)^3 & \Rightarrow & \varepsilon^2 & \varepsilon^4 & I & \Gamma_{C_3} \\ & (\varepsilon^2)^4 & (\varepsilon^2)^5 & (\varepsilon^2)^6 & & \varepsilon^2 & \varepsilon^4 & I & \\ & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow & \\ & C_6^4 & C_6^5 & I & & C_3 & C_3^2 & I & \end{array}$$

This half corresponds to the right half assigned to Γ_{C_3} for the group $\mathbf{C}_3$. Since lemma 2 indicates $\mathbf{C}_3 \cong \mathbf{C}_6/\mathbf{C}_2$ for the present example, the scheme depicted above shows that Γ_{C_3} for the cyclic group $\mathbf{C}_3$ can be amplified into $\Gamma_{C_6^2}$ for the cyclic group $\mathbf{C}_6$. Note that the sets appearing in the left half of the above scheme ($\{C_6, C_6^4\}$, $\{C_6^2, C_6^5\}$, and $\{C_6^3, I\}$) are the set of cosets corresponding to the factor group $\mathbf{C}_6/\mathbf{C}_2$.

2.3 Number-theoretic vectors

Since each irreducible representation of a cyclic group $\mathbf{C}_n$ is of degree 1, the following lemma is easily obtained.

Lemma 3. *Each irreducible representation of a cyclic group* $\mathbf{C}_n$ *appears once in the regular representation* $\mathbf{C}_n(/\mathbf{C}_1)$, i.e.,

$$\mathbf{C}_n(/\mathbf{C}_1) = \Gamma_{C_n} + \Gamma_{C_n^2} + \cdots + \Gamma_{C_n^n}. \tag{21}$$

Let $\tilde{\Gamma}_d = \tilde{\Gamma}_{n/(n/d)}$ be a row vector selected from a **Q**-conjugacy character table. The definition of **Q**-conjugacy characters [31] and the amplificative equivalence defined in Theorem 2 give

$$\tilde{\Gamma}_d = \sum_{(r,n/d)=1} \tilde{\Gamma}_{C_n^{rd}} = \sum_{(r,n/d)=1} \tilde{\Gamma}_{C_{n/d}^r}. \tag{22}$$

Thereby, Eq. (21) can be transformed into

$$\mathbf{C}_n(/\mathbf{C}_1) = \sum^{d|n} \sum_{(r,n/d)=1} \tilde{\Gamma}_{C_{n/d}^r} = \sum^{d|n} \tilde{\Gamma}_d, \tag{23}$$

where the symbol $d|n$ shows that the summation is over all divisors d of n. Note that all of the elements of $\mathbf{C}_n(/\mathbf{C}_1)$ and $\tilde{\Gamma}_d$ are integers. The amplificative equivalence defined in Theorem 1 combined with Eq. (23) gives a lemma.

Lemma 4. *The markaracter* $\mathbf{C}_n(/\mathbf{C}_d)$ *is represented by a sum of* **Q**-*conjugacy characters as follows*:

$$\mathbf{C}_n(/\mathbf{C}_d) \overset{\text{amp}}{=} \mathbf{C}_{n/d}(/\mathbf{C}_1) = \sum^{d'|\frac{n}{d}} \tilde{\Gamma}_{d'}. \tag{24}$$

Let us now consider the character (fixed-point vector) of each coset representation. For simplicity, the symbol for the coset representation is also used to designate the corresponding character. In light of this convention, let us now consider row vectors:

$$\boldsymbol{g}(\tilde{n}) = \mathbf{C}_n(/\mathbf{C}_d) \overset{\text{amp}}{=} \mathbf{C}_{\tilde{n}}(/\mathbf{C}_1) \tag{25}$$

$$\boldsymbol{f}(\tilde{n}) = \tilde{\Gamma}_{n/\tilde{n}}, \tag{26}$$

where $\tilde{n} = n/d$. It should be noted that the row vector $\boldsymbol{g}(\tilde{n})$ is the row of the corresponding markaracter table, while the row vector $\boldsymbol{f}(\tilde{n})$ is the row of the corresponding **Q**-conjugacy character table. Since each element of $\boldsymbol{g}(\tilde{n})$ and of $\boldsymbol{f}(\tilde{n})$ is a number-theoretical function, we call these vectors number-theoretical vectors. In light of Lemma 4, Theorem 1.22 of Ref. [33] for number-theoretical functions is easily extended so as to deal with number-theoretical vectors.

Theorem 3. *Since Eq.* (24) *gives*

$$\boldsymbol{g}(\tilde{n}) = \sum^{d'|\tilde{n}} \boldsymbol{f}(d') \tag{27}$$

we have

$$\boldsymbol{f}(\tilde{n}) = \sum^{d'|\tilde{n}} \mu\left(\frac{\tilde{n}}{d'}\right) \boldsymbol{g}(d'), \tag{28}$$

where $\mu\left(\frac{\tilde{n}}{d'}\right)$ *denotes the number-theoretical Möbius function.*

Theorem 3 is equivalent to the one described in Chapter 13.1 of Ref. [34], though the latter implies that the characters $1_{\mathbf{C}_d}^{\mathbf{C}_n}$ used in place of $\boldsymbol{g}(\tilde{n})$ are associated with conjugacy classes. On the other hand, the vectors $\boldsymbol{f}(\tilde{n})$ and $\boldsymbol{g}(\tilde{n})$ correspond to **Q**-conjugacy classes in the present treatment. In other words, the characters $1_{\mathbf{C}_d}^{\mathbf{C}_n}$ are class functions, while the vectors $\boldsymbol{f}(\tilde{n})$ and $\boldsymbol{g}(\tilde{n})$ are dominant-class (or **Q**-conjugacy-class) functions. This means that the **Q**-conjugacy character table and the markaracter table of a cyclic group C_n are square matrices of the same size. Note that the vectors $\boldsymbol{f}(\tilde{n})$ are the row vectors of the **Q**-conjugacy character table of a cyclic group $\mathbf{C}_n$ and the vectors $\boldsymbol{g}(\tilde{n})$ are the row vectors of the markaracter table of $\mathbf{C}_n$.

From the viewpoint of the present series of works, Theorem 3 can be regarded as showing the interconversion between a markaracter table and a **Q**-conjugacy character table for a cyclic group. For the purpose of describing this interconversion more clearly, we here use a Möbius function $\mu(d_i, d_j)$ as follows:

$$\mu(d_i, d_j) = \begin{cases} (-1)^r & \text{if } d_j/d_i \text{ is a product of } r \text{ of} \\ & \qquad \text{different prime numbers} \\ 0 & \text{if } d_j/d_i \text{ is a multiplier of the} \\ & \qquad \text{square of a prime number} \\ 0 & \text{if } d_j/d_i \text{ is not an integer.} \end{cases} \tag{29}$$

The usual number-theoretical Möbius function is slightly extended to cover rational numbers; thus, $\mu(k)$ (for $k = d_j/d_i$) is the usual number-theoretical Möbius function if k is an integer and is equal to zero otherwise [35]. Thereby, Theorem 3 is rewritten in terms of a matrix expression in the following corollary.

Corollary 1. *Let D be the* **Q**-*conjugacy character table of a cyclic group* $\mathbf{C}_n$. *Let M be the matrix derived by inversing the alignment of the rows from the markaracter table of* $\mathbf{C}_n$:

$$D = \begin{pmatrix} \boldsymbol{f}(d_1) \\ \boldsymbol{f}(d_2) \\ \vdots \\ \boldsymbol{f}(n) \end{pmatrix} = \begin{array}{c} \\ \tilde{\Gamma}_{d_1} \\ \tilde{\Gamma}_{d_2} \\ \vdots \\ \tilde{\Gamma}_n \end{array} \begin{array}{c} \begin{array}{cccc} \downarrow \mathbf{C}_{d_1} & \downarrow \mathbf{C}_{d_2} & & \downarrow \mathbf{C}_n \end{array} \\ \begin{pmatrix} \tilde{\gamma}_{11} & \tilde{\gamma}_{12} & \cdots & \tilde{\gamma}_{1s} \\ \tilde{\gamma}_{21} & \tilde{\gamma}_{22} & \cdots & \tilde{\gamma}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\gamma}_{s1} & \tilde{\gamma}_{s2} & \cdots & \tilde{\gamma}_{ss} \end{pmatrix} \end{array} \tag{30}$$

$$M = \begin{pmatrix} g(d_1) \\ g(d_2) \\ \vdots \\ g(n) \end{pmatrix} = \begin{array}{c} \\ \mathbf{C}_n(/\mathbf{C}_n) \\ \vdots \\ \mathbf{C}_n(/\mathbf{C}_{d_2}) \\ \mathbf{C}_n(/\mathbf{C}_{d_1}) \end{array} \begin{array}{c} \begin{array}{cccc} \downarrow \mathbf{C}_{d_1} & \downarrow \mathbf{C}_{d_2} & & \downarrow \mathbf{C}_n \end{array} \\ \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ m_{s-1\,1} & m_{s-1\,2} & \cdots & m_{s-1\,s} \\ m_{s1} & m_{s2} & \cdots & m_{ss} \end{pmatrix} \end{array}, \tag{31}$$

where the integers $d_1, d_2, \ldots$ represent the divisors of n and are aligned from small to large. Let us construct an interconversion matrix:

$$W = \begin{pmatrix} \mu(d_1, d_1) & \mu(d_2, d_1) & \cdots & \mu(n, d_1) \\ \mu(d_1, d_2) & \mu(d_2, d_2) & \cdots & \mu(n, d_2) \\ \vdots & \vdots & \ddots & \vdots \\ \mu(d_1, n) & \mu(d_2, n) & \cdots & \mu(n, n) \end{pmatrix}, \tag{32}$$

where the symbol $\mu(d_i, d_j)$ denotes the extended Möbius function defined by Eq. (29). *Then, D is obtaind from M by means of the following equation*:

$$D = WM. \tag{33}$$

The interconversion matrix W is easily shown to be a lower-triangular matrix in which all diagonal elements are equal to unity.

Example 3. The point group $\mathbf{C}_6$ has a markaracter table, each row of which is expressed by

$$g(6) = (6, 0, 0, 0)$$
$$g(3) = (3, 3, 0, 0)$$
$$g(2) = (2, 0, 2, 0)$$
$$g(1) = (1, 1, 1, 1).$$

Equations for this case (Eq. 27) are calculated to be

$$g(6) = f(1) + f(2) + f(3) + f(6)$$
$$g(3) = f(1) + f(3)$$
$$g(2) = f(1) + f(2)$$
$$g(1) = f(1).$$

In light of Theorem 3, we have

$$f(1) = f(1) = (1, 1, 1, 1)$$
$$f(2) = \mu\left(\frac{2}{1}\right) g(1) + \mu\left(\frac{2}{2}\right) g(2) = -g(1) + g(2) = (1, -1, 1, -1)$$
$$f(3) = \mu\left(\frac{3}{1}\right) g(1) + \mu\left(\frac{3}{3}\right) g(2) = -g(1) + g(3) = (2, 2, -1, -1)$$
$$f(6) = \mu\left(\frac{6}{1}\right) g(1) + \mu\left(\frac{6}{2}\right) g(2) + \mu\left(\frac{6}{3}\right) g(3) + \mu\left(\frac{6}{6}\right) g(6) = g(1) - g(2) - g(3) + g(6) = (2, -2, -1, 1).$$

The resulting vectors, $f(1)$, $f(2)$, $f(3)$, and $f(6)$, are the row vectors of the $\mathbf{Q}$-conjugacy character table of $\mathbf{C}_6$. Thus, these data can be summarized to give a matrix expression,

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 2 & 2 & -1 & -1 \\ 2 & -2 & -1 & 1 \end{pmatrix} = \begin{array}{c} A \\ B \\ E_2 \\ E_1 \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 3 & 3 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{pmatrix}, \tag{34}$$

where the matrix in the left-hand side represents the $\mathbf{Q}$-conjugacy character table of $\mathbf{C}_6$.

3 Characteristic monomials

3.1 The first and the last column of a characteristic monomial table for a cyclic group

From Eq. (33) of Corollary 1, we easily obtain the following equation:

$$DM^{-1} = W. \tag{35}$$

Since M is an alternative form of a markaracter table, the left-hand side of Eq. (35) indicates that this case is a special case of Theorem 1 of the preceding paper. In other words, the matrix W appearing in the right-hand side of Eq. (35) is the multiplicity matrix for the subduction of a $\mathbf{Q}$-conjugacy representation of $\mathbf{C}_n$ into $\mathbf{C}_n$, i.e., $\Gamma_{n/\tilde{n}} \downarrow \mathbf{C}_n$. Thus, each element of a row of W indicates the power of a dummy variable $s_{d'}$ for the corresponding divisor $d' = n/d_i$. By using the notation of Theorem 3, we define a characteristic monomial for $\Gamma_{n/\tilde{n}} \downarrow \mathbf{C}_n$ as follows.

$$Z(\Gamma_{n/\tilde{n}} \downarrow \mathbf{C}_n; s_{d'}) = \prod^{d'|\tilde{n}} s_{d'}^{\mu(\tilde{n}/d')}. \tag{36}$$

The dummy variables derived by Eq. (36) construct the last column of a characteristic monomial table for a cyclic group.

On the other hand, the subduction $\Gamma_{n/\tilde{n}} \downarrow \mathbf{C}_1$ obviously corresponds to the following monomial:

$$Z(\Gamma_{n/\tilde{n}} \downarrow \mathbf{C}_1; s_{d'}) = s_1^{\varphi(\tilde{n})}, \tag{37}$$

where $\varphi(\tilde{n})$ is the Euler function. The dummy variables derived by Eq. (37) construct the first column of a characteristic monomial table for a cyclic group

Lemma 5. *Each monomial represented by Eq.* (36) *contains dummy variables with the power of* -1, 0, *or* 1 *because of the nature of the extended Möbius function. Each monomial represented by Eq.* (37) *contains dummy variables with the power of a natural number*.

3.2 Characteristic monomial tables for cyclic groups of prime-number order

By means of Eqs. (36) and (37), we are able to construct the characteristic monomial tables for cyclic groups having the order of a prime number p. The point group $\mathbf{C}_p$ has a markaracter table, each row of which is expressed by

$$\boldsymbol{g}(p) = (p, 0)$$
$$\boldsymbol{g}(1) = (1, 1).$$

Equations for this case (Eq. 27) are calculated to be

$$\boldsymbol{g}(p) = \boldsymbol{f}(1) + \boldsymbol{f}(p)$$
$$\boldsymbol{g}(1) = \boldsymbol{f}(1)\ .$$

Thereby, we have the following relationship:

$$\boldsymbol{f}(1) = \boldsymbol{g}(1)$$
$$\boldsymbol{f}(p) = \mu\left(\frac{p}{1}\right)\boldsymbol{g}(1) + \mu\left(\frac{p}{p}\right)\boldsymbol{g}(p) = -\boldsymbol{g}(1) + \boldsymbol{g}(p), \tag{38}$$

which produces the rows of the **Q**-conjugacy character table of $\mathbf{C}_p$.

$$\boldsymbol{f}(p) = (p-1, 1)$$
$$\boldsymbol{f}(1) = (1, -1).$$

The transformation described above can be shown more clearly in terms of Eq. (32). Thus, we have the matrix W for this case:

$$W = \begin{pmatrix} \mu(1,1) & \mu(p,1) \\ \mu(1,p) & \mu(p,p) \end{pmatrix} = \begin{pmatrix} \mu(1) & 0 \\ \mu(p) & \mu(1) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \tag{39}$$

This matrix gives the multiplicities that are the powers of such dummy variables. As a result, Eq. (36) gives characteristic monomials for the group $\mathbf{C}_p$:

$$\begin{pmatrix} \boldsymbol{f}(1) \\ \boldsymbol{f}(p) \end{pmatrix} = \begin{matrix} \Gamma_p \\ \Gamma_1 \end{matrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{g}(1) \\ \boldsymbol{g}(p) \end{pmatrix} \begin{matrix} \cdots & s_1 \\ \cdots & s_1^{-1}s_p \end{matrix}. \tag{40}$$

This result combined with the one derived from Eq. (37) gives the characteristic monomial table for $\mathbf{C}_p$ (Table 5).
Example 4. The point group $\mathbf{C}_2$ has a markaracter table, each row of which is expressed by

$$\boldsymbol{g}(2) = (2, 0)$$
$$\boldsymbol{g}(1) = (1, 1).$$

Equations for this case (Eq. 27) are calculated to be

$$\boldsymbol{g}(2) = \boldsymbol{f}(1) + \boldsymbol{f}(2)$$
$$\boldsymbol{g}(1) = \boldsymbol{f}(1).$$

Hence, we have

$$\boldsymbol{f}(1) = \boldsymbol{g}(1) = (1, 1)$$
$$\boldsymbol{f}(2) = \mu\left(\frac{2}{1}\right)\boldsymbol{g}(1) + \mu\left(\frac{2}{2}\right)\boldsymbol{g}(2)$$
$$= -\boldsymbol{g}(1) + \boldsymbol{g}(2) = (1, -1).$$

These data can be summarized into an matrix expression. Since the number-theoretical vector $\boldsymbol{g}(\tilde{n})$ corresponds to the coset representation $\mathbf{C}_{\tilde{n}}(/\mathbf{C}_1)$, we select a dummy variable $s_{\tilde{n}}$ which is equal to the size of the orbit governed by $\mathbf{C}_{\tilde{n}}(/\mathbf{C}_1)$. The 2×2 matrix represents the multiplicities of the $\boldsymbol{g}(\tilde{n})$ vectors [31]. The multiplicities are the powers of such dummy variables which give a monomial for each **Q**-conjugacy representation. As a result, Eq. (36) gives dummy variables for the group $\mathbf{C}_2$

$$\begin{pmatrix} \boldsymbol{f}(1) \\ \boldsymbol{f}(2) \end{pmatrix} = \begin{matrix} A \\ B \end{matrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{g}(1) \\ \boldsymbol{g}(2) \end{pmatrix} \begin{matrix} \cdots & s_1 \\ \cdots & s_1^{-1}s_2. \end{matrix} \tag{41}$$

This result combined with the one derived from Eq. (37) gives the characteristic monomial table for $\mathbf{C}_2$ (Table 6).

Similarly, the point group $\mathbf{C}_3$ has a markaracter table shown in Table 7.

3.3 Characteristic monomial tables for cyclic groups

Let $\mathbf{C}_n$ be a cyclic group having $\mathbf{C}_u$ as a cyclic subgroup. Suppose we have obtained the **Q**-conjugacy character table of $\mathbf{C}_u$ denoted as $D_{\mathbf{C}_u}$ and the characteristic monomial table of $\mathbf{C}_u$, where each monomial of the last column is represented by

$$Z\left(\Gamma^{(u)}_{u/\tilde{u}}; s_{d'}\right), \tag{42}$$

where $\tilde{u}$ runs over all of the divisors of u. Note that the variable is obtained by applying Eq. (36) to the subgroup $\mathbf{C}_u$. Suppose, in addition, that we have the **Q**-conjugacy character table of $\mathbf{C}_n$ denoted as $D_{\mathbf{C}_n}$. Then, these data are capable of giving the characteristic monomial table of C_n.

From the table $D_{\mathbf{C}_n}$, we select the columns corresponding to the subgroup $\mathbf{C}_u$. As the result of this subduction, we have a matrix designated by the symbol $D_{\mathbf{C}_n\downarrow\mathbf{C}_u}$. This matrix is multiplied by the inverse of $D_{\mathbf{C}_u}$ as follows.

$$D_{\mathbf{C}_n\downarrow\mathbf{C}_u} D^{-1}_{\mathbf{C}_u} = Y, \tag{43}$$

where each row of the resulting matrix Y is represented by

$$Y_{n/\tilde{n}} = (y_1, \ldots, y_{u/\tilde{u}}, \ldots), \tag{44}$$

where $\tilde{u}$ runs over all of the divisors of u. Each element of $Y_{n/\tilde{n}}$ represents the multiplicity of $\Gamma^{(u)}_{u/\tilde{u}}$ that is a **Q**-conjugacy representation of $\mathbf{C}_u$. Thus, we have

Table 5. Characteristic monomials for $\mathbf{C}_p$

	$\downarrow \mathbf{C}_1$	$\downarrow \mathbf{C}_p$
Γ_p	s_1	s_1
Γ_1	s_1^{p-1}	$s_1^{-1}s_p$

Table 6. Characteristic monomials for $\mathbf{C}_2$

	$\downarrow \mathbf{C}_1$	$\downarrow \mathbf{C}_2$
A	s_1	s_1
B	s_1	$s_1^{-1}s_2$

$$\Gamma^{(n)}_{n/\tilde{n}} \downarrow \mathbf{C}_u = \sum^{\tilde{u}|u} y_{u/\tilde{u}} \Gamma^{(u)}_{u/\tilde{u}}, \tag{45}$$

where $\Gamma^{(n)}_{n/\tilde{n}}$ denotes each row of $D_{\mathbf{C}_n}$. According to Eq. (45), the corresponding characteristic monomial is defined by

$$Z\left(\Gamma^{(n)}_{n/\tilde{n}} \downarrow \mathbf{C}_u; s_{d'}\right) = \prod^{\tilde{u}|u} \left[Z\left(\Gamma^{(u)}_{u\tilde{u}}; s_{d'}\right)\right]^{y_{u/\tilde{u}}} \tag{46}$$

by using Eq. (43). The variables in the right-hand side of Eq. (46) appear in the last column of the characteristic monomial table of $\mathbf{C}_u$ that is obtained by applying Eq. (36) to the subgroup $\mathbf{C}_u$.

The monomials obtained by Eq. (46) when $\tilde{n}$ runs over all of the divisors of n construct the $\downarrow \mathbf{C}_u$ column of the characteristic monomial table of $\mathbf{C}_n$.

Example 5. Let us revisit the point group $\mathbf{C}_6$. The results of Example 3 are summarized into a matrix expression:

$$\begin{pmatrix} \boldsymbol{f}(1) \\ \boldsymbol{f}(2) \\ \boldsymbol{f}(3) \\ \boldsymbol{f}(6) \end{pmatrix} = \begin{array}{c} A \\ B \\ E_2 \\ E_1 \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} \boldsymbol{g}(1) \\ \boldsymbol{g}(2) \\ \boldsymbol{g}(3) \\ \boldsymbol{g}(6) \end{pmatrix} \begin{array}{cc} \cdot & \underline{\downarrow \mathbf{C}_6} \\ & s_1 \\ \cdots & s_1^{-1}s_2 \\ \cdots & s_1^{-1}s_3 \\ \cdots & s_1 s_2^{-1} s_3^{-1} s_6 \end{array}. \tag{47}$$

The 4×4 matrix represents the multiplicities of the $\boldsymbol{g}(\tilde{n})$ vectors [31]. The multiplicities are the powers of such dummy variables which give a monomial for each **Q**-conjugacy representation. The monomials construct the column of $\downarrow \mathbf{C}_6$.

The monomials for the column of $\downarrow \mathbf{C}_2$ are obtained by a two-step procedure. First, we select the $\downarrow \mathbf{C}_1$ and $\downarrow \mathbf{C}_2$ columns from the **Q**-conjugacy character table of $\mathbf{C}_6$ to form a 4×2 matrix, which is multiplied by the inverse ($D^{-1}_{\mathbf{C}_2}$)of the **Q**-conjugacy character table of $\mathbf{C}_2$. The resulting 4×2 matrix contains the multiplicities of **Q**-conjugacy characters of $\mathbf{C}_2$ as shown after the dotted lines. Second, we take monomials from Table 6 according to the multiplicities and multiply them to obtain the characteristic monomials for the column of $\downarrow \mathbf{C}_2$.

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 2 \\ 2 & -2 \end{pmatrix} \overset{D^{-1}_{\mathbf{C}_2}}{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}$$

$$= \begin{array}{c} \\ A \\ B \\ E_2 \\ E_1 \end{array} \begin{array}{c} \begin{array}{cc} A & E \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} \end{array} \begin{array}{ccccc} & \mathbf{C}_2 & & \downarrow \mathbf{C}_1 & \downarrow \mathbf{C}_2 \\ \cdots & A & \cdots & s_1 & s_1 \\ \cdots & E & \cdots & s_1 & s_1^{-1}s_2 \\ \cdots & 2A & \cdots & s_1^2 & s_1^2 \\ \cdots & 2E & \cdots & s_1^2 & s_1^{-2}s_2^2. \end{array} \tag{48}$$

The monomials for the column of $\downarrow \mathbf{C}_3$ are also obtained by a two-step procedure. We use the inverse ($D^{-1}_{\mathbf{C}_3}$) of the **Q**-conjugacy character table of $\mathbf{C}_3$ and the data of Table 7

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 2 & -1 \\ 2 & -1 \end{pmatrix} \overset{D^{-1}_{\mathbf{C}_3}}{\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}}$$

$$= \begin{array}{c} \\ A \\ B \\ E_2 \\ E_1 \end{array} \begin{array}{c} \begin{array}{cc} A & E \end{array} \\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \end{array} \begin{array}{ccccc} & \mathbf{C}_3 & & \downarrow \mathbf{C}_1 & \downarrow \mathbf{C}_3 \\ \cdots & A & \cdots & s_1 & s_1 \\ \cdots & A & \cdots & s_1 & s_1 \\ \cdots & E & \cdots & s_1^2 & s_1^{-1}s_3 \\ \cdots & E & \cdots & s_1^2 & s_1^{-1}s_3. \end{array} \tag{49}$$

The resulting columns of monomials are collected to give a characteristic monomial table for $\mathbf{C}_6$ (Table 8).

3.4 Characteristic monomial tables for finite groups

Let $\mathbf{G}$ be a cyclic group having $\mathbf{C}_u$ as a cyclic subgroup. The characteristic monomial table of $\mathbf{C}_u$ contains the last column that consists of monomials represented by Eq. (42). The **Q**-conjugacy characters of $\mathbf{G}$ ($\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$, ... $\tilde{\Gamma}_s$) are regarded as the row vectors of the **Q**-conjugacy character table $D_{\mathbf{G}}$ ($s \times s$ matrix). We select the columns corresponding to the subgroup $\mathbf{C}_u$ to produce an $s \times r$ matrix designated by the symbol $D_{\mathbf{G}\downarrow\mathbf{C}_u}$. This matrix is multiplied by the inverse of $D_{\mathbf{C}_u}$ as follows.

$$D_{\mathbf{G}\downarrow\mathbf{C}_u} D^{-1}_{\mathbf{C}_u} = X, \tag{50}$$

where each row of the resulting matrix X is represented by

$$X_i = (x_1, \ldots, x_{u/\tilde{u}}, \ldots), \tag{51}$$

for $i = 1, 2, \ldots, s$, Each element of X_i represents the multiplicity of $\Gamma^{(u)}_{u/\tilde{u}}$ that is a **Q**-conjugacy representation of $\mathbf{C}_u$. Thus, we have

$$\Gamma_i \downarrow \mathbf{C}_u = \sum^{\tilde{u}|u} x_{u/\tilde{u}} \Gamma^{(u)}_{u/\tilde{u}}, \tag{52}$$

Table 7. Characteristic monomials for $\mathbf{C}_3$

	$\downarrow \mathbf{C}_1$	$\downarrow \mathbf{C}_3$
A	s_1	s_1
E	s_1^2	$s_1^{-1}s_3$

Table 8. Characteristic monomials for $\mathbf{C}_6$

$\mathbf{C}_6$	$\downarrow \mathbf{C}_1$	$\downarrow \mathbf{C}_2$	$\downarrow \mathbf{C}_3$	$\downarrow \mathbf{C}_6$
A	s_1	s_1	s_1	s_1
B	s_1	$s_1^{-1}s_2$	s_1	$s_1^{-1}s_2$
E_2	s_1^2	s_1^2	$s_1^{-1}s_3$	$s_1^{-1}s_3$
E_1	s_1^2	$s_1^{-2}s_2^2$	$s_1^{-1}s_3$	$s_1 s_2^{-1} s_3^{-1} s_6$

where Γ_i denotes the $\mathbf{Q}$-conjugacy representation corresponding to each row $(\tilde{\Gamma}_i)$ of $D_{\mathbf{G}}$. Hence, the corresponding characteristic monomial is defined by

$$Z(\Gamma_i \downarrow \mathbf{C}_u; s_{d'}) = \prod^{\tilde{u}|u} \left[Z\left(\Gamma^{(u)}_{u/\tilde{u}}; s_{d'}\right)\right]^{x_{u/\tilde{u}}} \tag{53}$$

for $i = 1, 2, \ldots, s$ by using Eq. (50).

The monomials obtained by Eq. (53) when $\tilde{n}$ runs over all of the divisors of n construct the $\downarrow \mathbf{C}_u$ column of the characteristic monomial table of $\mathbf{C}_n$.

Example 6. Let us examine the point group $\mathbf{T}$, which has a $\mathbf{Q}$-conjugacy character table:

$$D_{\mathbf{T}} = \begin{array}{c} \\ A \\ E \\ T \end{array} \!\!\! \begin{array}{c} \begin{array}{ccc} \downarrow \mathbf{C}_1 & \downarrow \mathbf{C}_2 & \downarrow \mathbf{C}_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ 3 & -1 & 0 \end{pmatrix} \end{array}. \tag{54}$$

The monomials for the column of $\downarrow \mathbf{C}_2$ are obtained by a two-step procedure. First, we select the $\downarrow \mathbf{C}_1$ and $\downarrow \mathbf{C}_2$ columns from the $\mathbf{Q}$-conjugacy character table of $\mathbf{T}$ to form a 3×2 matrix, which is multiplied by the inverse $(D^{-1}_{\mathbf{C}_2})$ of the $\mathbf{Q}$-conjugacy character table of $\mathbf{C}_2$. The resulting 3×2 matrix contains the multiplicities of $\mathbf{Q}$-conjugacy characters of $\mathbf{C}_2$ as shown after the dotted lines. Second, we take monomials from Table 6 according to the multiplicities and multiply them to obtain the characteristic monomials shown in the column of $\downarrow \mathbf{C}_2$.

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & -1 \end{pmatrix} \overset{D^{-1}_{\mathbf{C}_2}}{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}$$

$$= \begin{array}{c} \\ A \\ E \\ T \end{array} \!\!\! \begin{array}{c} \begin{array}{cc} A & E \end{array} \\ \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 2 \end{pmatrix} \end{array} \begin{array}{ccccc} & \mathbf{C}_2 & & \downarrow \mathbf{C}_1 & \downarrow \mathbf{C}_2 \\ \cdots & A & \cdots & s_1 & s_1 \\ \cdots & 2A & \cdots & s_1^2 & s_1^2 \\ \cdots & A + 2E & \cdots & s_1^3 & s_1^{-1} s_2^2. \end{array} \tag{55}$$

The monomials for the column of $\downarrow \mathbf{C}_3$ are also obtained by a two-step procedure. We use the inverse $(D^{-1}_{\mathbf{C}_3})$ of the $\mathbf{Q}$-conjugacy character table of $\mathbf{C}_3$ and the data of Table 7

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 3 & 0 \end{pmatrix} \overset{D^{-1}_{\mathbf{C}_3}}{\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}}$$

$$= \begin{array}{c} \\ A \\ E \\ T \end{array} \!\!\! \begin{array}{c} \begin{array}{cc} A & E \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array} \begin{array}{ccccc} & \mathbf{C}_3 & & \downarrow \mathbf{C}_1 & \downarrow \mathbf{C}_3 \\ \cdots & A & \cdots & s_1 & s_1 \\ \cdots & E & \cdots & s_1^2 & s_1^{-1} s_3 \\ \cdots & A + E & \cdots & s_1^3 & s_3. \end{array} \tag{56}$$

Table 9. Characteristic monomials for $\mathbf{T}$

$\mathbf{T}$	$\downarrow \mathbf{C}_1$	$\downarrow \mathbf{C}_2$	$\downarrow \mathbf{C}_3$
A	s_1	s_1	s_1
E	s_1^2	s_1^2	$s_1^{-1} s_3$
T	s_1^3	$s_1^{-1} s_2^2$	s_3

The resulting columns of monomials are collected to give a characteristic monomial table for $\mathbf{T}$ (Table 9).

Example 7. Let us examine the point group $\mathbf{T}_h$, which has a $\mathbf{Q}$-conjugacy character table:

$$D_{\mathbf{T}_h} = \begin{array}{c} \\ A_g \\ A_u \\ E_g \\ E_u \\ T_g \\ T_g \end{array} \!\!\! \begin{array}{c} \begin{array}{cccccc} \downarrow \mathbf{C}_1 & \downarrow \mathbf{C}_2 & \downarrow \mathbf{C}_s & \downarrow \mathbf{C}_i & \downarrow \mathbf{C}_3 & \downarrow \mathbf{S}_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 \\ 2 & 2 & 2 & 2 & -1 & -1 \\ 2 & 2 & -2 & -2 & -1 & 1 \\ 3 & -1 & -1 & 3 & 0 & 0 \\ 3 & -1 & 1 & -3 & 0 & 0 \end{pmatrix} \end{array}. \tag{57}$$

The monomials for the column of $\downarrow \mathbf{C}_2$ are obtained by a two-step procedure. First, we select the $\downarrow \mathbf{C}_1$ and $\downarrow \mathbf{C}_2$ columns from the $\mathbf{Q}$-conjugacy character table of $\mathbf{T}_h$ to form a 6×2 matrix, which is multiplied by the inverse $(D^{-1}_{\mathbf{C}_2})$ of the $\mathbf{Q}$-conjugacy character table of $\mathbf{C}_2$. The resulting 6×2 matrix contains the multiplicities of $\mathbf{Q}$-conjugacy characters of $\mathbf{C}_2$ as shown after the dotted lines. Second, we take monomials from Table 6 according to the multiplicities and multiply them to obtain the characteristic monomials shown in the column of $\downarrow \mathbf{C}_2$.

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 2 \\ 2 & 2 \\ 3 & -1 \\ 3 & -1 \end{pmatrix} \overset{D^{-1}_{\mathbf{C}_2}}{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}}$$

$$= \begin{array}{c} \\ A_g \\ A_u \\ E_g \\ E_u \\ T_g \\ T_u \end{array} \!\!\! \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 2 & 0 \\ 2 & 0 \\ 1 & 2 \\ 1 & 2 \end{pmatrix} \end{array} \begin{array}{cccc} & \mathbf{C}_2 & & \downarrow \mathbf{C}_2 \\ \cdots & A & \cdots & s_1 \\ \cdots & A & \cdots & s_1 \\ \cdots & 2A & \cdots & s_1^2 \\ \cdots & 2A & \cdots & s_1^2 \\ \cdots & A + 2B & \cdots & s_1^{-1} s_2^2 \\ \cdots & A + 2B & \cdots & s_1^{-1} s_2^2 \end{array} \tag{58}$$

We select the $\downarrow \mathbf{C}_1$ and $\downarrow \mathbf{C}_s$ columns from the $\mathbf{Q}$-conjugacy character table of $\mathbf{T}_h$ to form a 6×2 matrix, from which we obtain the multiplicities of $\mathbf{Q}$-conjugacy characters:

$$\begin{pmatrix}1&1\\1&-1\\2&2\\2&-2\\3&-1\\3&1\end{pmatrix}\overset{D_{\mathbf{C}_s}^{-1}}{\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&-\frac{1}{2}\end{pmatrix}}$$

$$=\begin{array}{c}\\A_g\\A_u\\E_g\\E_u\\T_g\\T_u\end{array}\begin{array}{c}\begin{array}{cc}A'&A''\end{array}\\\begin{pmatrix}1&0\\0&1\\2&0\\0&2\\1&2\\2&1\end{pmatrix}\end{array}\begin{array}{ccccc}&&\mathbf{C}_s&&\underline{\downarrow\mathbf{C}_s}\\\cdots&&A'&\cdots&s_1\\\cdots&&A''&\cdots&s_1^{-1}s_2\\\cdots&&2A'&\cdots&s_1^2\\\cdots&&2A''&\cdots&s_1^{-2}s_2^2\\\cdots&&A'+2A''&\cdots&s_1^{-1}s_2^2\\\cdots&&2A'+A''&\cdots&s_2.\end{array}\tag{59}$$

We select the $\downarrow\mathbf{C}_1$ and $\downarrow\mathbf{C}_i$ columns from the **Q**-conjugacy character table of $\mathbf{T}_h$ to form a 6×2 matrix, from which we obtain the multiplicities of **Q**-conjugacy characters:

$$\begin{pmatrix}1&1\\1&-1\\2&2\\2&-2\\3&3\\3&-3\end{pmatrix}\overset{D_{\mathbf{C}_i}^{-1}}{\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&-\frac{1}{2}\end{pmatrix}}$$

$$=\begin{array}{c}\\A_g\\A_u\\E_g\\E_u\\T_g\\T_u\end{array}\begin{array}{c}\begin{array}{cc}A_g&A_u\end{array}\\\begin{pmatrix}1&0\\0&1\\2&0\\0&2\\3&0\\0&3\end{pmatrix}\end{array}\begin{array}{ccccc}&&\mathbf{C}_i&&\underline{\downarrow\mathbf{C}_i}\\\cdots&&A_g&\cdots&s_1\\\cdots&&A_u&\cdots&s_1^{-1}s_2\\\cdots&&2A_g&\cdots&s_1^2\\\cdots&&2A_u&\cdots&s_1^{-2}s_2^2\\\cdots&&3A_g&\cdots&s_1^3\\\cdots&&3A_u&\cdots&s_1^{-3}s_2^3.\end{array}\tag{60}$$

The monomials for the column of $\downarrow\mathbf{C}_3$ are also obtained by a two-step procedure. We use the inverse ($D_{\mathbf{C}_3}^{-1}$) of the **Q**-conjugacy character table of $\mathbf{C}_3$ and the data of Table 7

$$\begin{pmatrix}1&1\\1&1\\2&-1\\2&-1\\3&0\\3&0\end{pmatrix}\overset{D_{\mathbf{C}_3}^{-1}}{\begin{pmatrix}\frac{1}{3}&\frac{1}{3}\\\frac{2}{3}&-\frac{1}{3}\end{pmatrix}}$$

$$=\begin{array}{c}\\A_g\\A_u\\E_g\\E_u\\T_g\\T_u\end{array}\begin{array}{c}\begin{array}{cc}A&E\end{array}\\\begin{pmatrix}1&0\\1&0\\0&1\\0&1\\1&1\\1&1\end{pmatrix}\end{array}\begin{array}{ccccc}&&\mathbf{C}_3&&\underline{\downarrow\mathbf{C}_3}\\\cdots&&A&\cdots&s_1\\\cdots&&A&\cdots&s_1\\\cdots&&E&\cdots&s_1^{-1}s_3\\\cdots&&E&\cdots&s_1^{-1}s_3\\\cdots&&A+E&\cdots&s_3\\\cdots&&A+E&\cdots&s_3.\end{array}\tag{61}$$

The monomials for the column of $\downarrow\mathbf{S}_6$ are also obtained by a two-step procedure. We use the inverse ($D_{\mathbf{S}_6}^{-1}$) of the **Q**-conjugacy character table of $\mathbf{S}_6$ and the data of Table 7. Note that the group $\mathbf{S}_6$ is isomorphic to the group $\mathbf{C}_6$.

$$\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\2&2&-1&-1\\2&-2&-1&1\\3&3&0&0\\3&-3&0&0\end{pmatrix}\overset{D_{\mathbf{C}_3}^{-1}}{\begin{pmatrix}\frac{1}{6}&\frac{1}{6}&\frac{1}{6}&\frac{1}{6}\\\frac{1}{6}&-\frac{1}{6}&-\frac{1}{6}&\frac{1}{6}\\\frac{1}{3}&\frac{1}{3}&-\frac{1}{6}&-\frac{1}{6}\\\frac{1}{3}&-\frac{1}{3}&\frac{1}{6}&-\frac{1}{6}\end{pmatrix}}$$

$$=\begin{array}{c}\\A_g\\A_u\\E_g\\E_u\\T_g\\T_u\end{array}\begin{array}{c}\begin{array}{cccc}A&B&E_1&E_2\end{array}\\\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&1\\0&1&1&0\end{pmatrix}\end{array}\begin{array}{ccccc}&&\mathbf{S}_6&&\underline{\downarrow\mathbf{S}_6}\\\cdots&&A&\cdots&s_1\\\cdots&&B&\cdots&s_1^{-1}s_2\\\cdots&&E_2&\cdots&s_1^{-1}s_3\\\cdots&&E_1&\cdots&s_1s_2^{-1}s_3^{-1}s_6\\\cdots&&A+E_2&\cdots&s_3\\\cdots&&B+E_1&\cdots&s_3^{-1}s_6.\end{array}\tag{62}$$

The resulting columns of monomials are collected to give a characteristic monomial table for $\mathbf{T}_h$ (Table 10).

The discussions described in this and the preceding sections permit us to extend Lemma 5 for a special case into a more general case to give a theorem.

Theorem 4. *Each characteristic monomial contains dummy variables with the power of an integer.*

4 Combinatorial enumeration

Suppose that a skeleton of symmetry $\mathbf{G}$ has a set of positions which are associated with a permutation representation $\mathbf{P}$. The permutation representation is converted into the corresponding matrix representation. The latter gives a fixed-point vector (FPV), each element of which is the number of points fixed under every subgroup action. The FPV can be regarded as a **Q**-conjugacy character which is multiplied by the inverse of the **Q**-conjugacy character table of $\mathbf{G}$ (i.e., $D_{\mathbf{G}}^{-1}$) to give the multiplicities of **Q**-conjugacy characters. These multiplicities correspond to the multiplicities (α_i) of **Q**-conjugacy representations, Γ_i $(i=1,2,\ldots)$, which

Table 10. Characteristic monomials for $\mathbf{T}_h$

$\mathrm{T_h}$	$\downarrow\mathbf{C}_1$	$\downarrow\mathbf{C}_2$	$\downarrow\mathbf{C}_3$	$\downarrow\mathbf{C}_i$	$\downarrow\mathbf{C}_3$	$\downarrow\mathbf{S}_6$
A_g	s_1	s_1	s_1	s_1	s_1	s_1
A_u	s_1	s_1	$s_1^{-1}s_2$	$s_1^{-1}s_2$	s_1	$s_1^{-1}s_2$
E_g	s_1^2	s_1^2	s_1^2	s_1^2	$s_1^{-1}s_3$	$s_1^{-1}s_3$
E_u	s_1^2	s_1^2	$s_1^{-2}s_2^2$	$s_1^{-2}s_2^2$	$s_1^{-1}s_3$	$s_1s_2^{-1}s_3^{-1}s_6$
T_g	s_1^3	$s_1^{-1}s_2^2$	$s_1^{-1}s_2^2$	s_1^3	s_3	s_3
T_u	s_1^3	$s_1^{-1}s_2^2$	s_1s_2	$s_1^{-3}s_2^3$	s_3	$s_3^{-1}s_6$
N_u	$\frac{1}{24}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{3}$	$\frac{1}{3}$

represent a subdivision of the positions under the action of the group $\mathbf{G}$. This is symbolically represented by the following equation.

$$\mathbf{P} = \sum_i \alpha_i \Gamma_i. \tag{63}$$

Since the subductions of every representation Γ_i are assigned to the characteristic monomials represented by Eq. (53), these are multiplied in accord with Eq. (63) to define an subduced cycle index (SCI):

$$\mathrm{SCI}(\mathbf{G} \downarrow \mathbf{C}_u; s'_d) = \prod_i \left[Z(\Gamma_i \downarrow \mathbf{C}_u; s'_d)\right]^{\alpha_i}, \tag{64}$$

where the cyclic subgroup $\mathbf{C}_u$ is tentatively fixed. In a similar way to definition 4 of Ref. [36], we have the definition of a cycle index (CI) on the basis of Eq. (64):

$$\mathrm{CI}(\mathbf{G}; s'_d) = \sum_u N_u \prod_i \left[Z(\Gamma_i \downarrow \mathbf{C}_u; s'_d)\right]^{\alpha_i}, \tag{65}$$

where we place

$$N_u = \frac{\varphi(|\mathbf{C}_u|)}{|N_{\mathbf{G}}(\mathbf{C}_u)|} \tag{66}$$

in light of Eq. (54) of Ref. [29]. The CI based on Eq. (65) is capable of combinatorial enumeration in place of the CI obtained alternatively in Corollary 1 described in the preceding paper. Hence, this is restated as a theorem by using the CI based on Eq. (65).

Theorem 5. *Suppose that η_γ of ligands X_γ ($\gamma = 1, 2, \ldots, v$) are selected from a set of ligands*

$$\mathbf{X} = \{X_1, X_2, \ldots, X_v\}, \tag{67}$$

where we have a partition:

$$[\eta] = \eta_1 + \eta_2 + \cdots + \eta_v = n. \tag{68}$$

They are placed on n of the positions in a skeleton to give isomers with the weight (molecular formula)

$$W_\eta = \prod_{\gamma=1}^{v} X_\gamma^{\eta_\gamma}. \tag{69}$$

A generating function for the total number A_η of isomers with the weight W_η is represented by

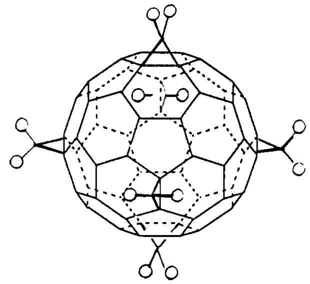
$$\sum_\eta A_\eta W_\eta = \mathrm{CI}\ (\mathbf{G}; s_{d'}), \tag{70}$$

where

$$s_{d'} = \sum_{\gamma=1}^{v} X_\gamma^{d'}. \tag{71}$$

This theorem gives enumeration results equivalent to those of Pólya's theorem, though the definitions of the CI are different between the two theorems.

Example 8. [60]Fullerene (C_{60}) of $\mathbf{I}_h$ symmetry gives a derivative of $\mathbf{T}_h$ symmetry. The symmetrical properties of the derivative have been disscussed in terms of the subduction-of-coset representation approach [37], where its six addents have been shown to belong to the $\mathbf{T}_h(/\mathbf{C}_{2v})$ orbit. Let us here consider the $\mathbf{T}_h$ derivative **1** as a mother skeleton, in which each of the six addents is a *gem*-dibromomethylene moiety, in which each circle denotes a bromine atom. For the purpose of comprehension of the symmetrical features of $\mathbf{T}_h$, the $\mathbf{T}_h$ skeleton is schematically represented by **1′** in which *gem*-dibromomethylene moieties are expressed by thick lines.



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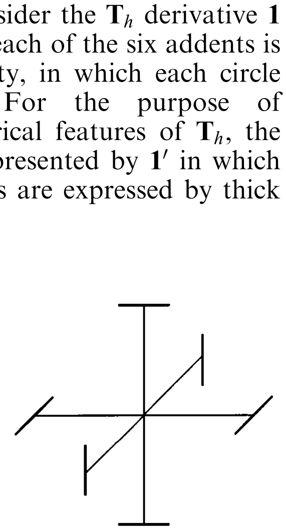
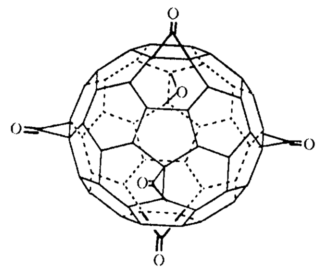


1 **1′**

Complete hydrolysis of the *gem*-dibromomethylene groups yields derivative **2** whose carbonyl groups are denoted by $=$O. For simplicity, each resulting carbonyl group is expressed by a thick line with a circle as shown in **2′**.

Suppose that a set of several *gem*-dibromomethylene moieties is partially converted into carbonyl moieties to produce a derivative having a subsymmetry of $\mathbf{T}_h$.



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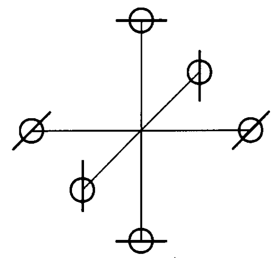

2 **2′**

The $\mathbf{T}_h(/\mathbf{C}_{2v})$ orbit gives $(6, 2, 4, 0, 0, 0)$ as an FPV, which is multiplied by the inverse of the $\mathbf{Q}$-conjugacy character table of the $\mathbf{T}_h$ group, i.e., $D_{\mathbf{T}_h}^{-1}$ derived from Eq. (57). It follows that

$$(6, 2, 4, 0, 0, 0) \begin{pmatrix} \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{24} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{8} & -\frac{1}{8} & \frac{1}{8} \\ \frac{1}{24} & -\frac{1}{24} & \frac{1}{24} & -\frac{1}{24} & \frac{1}{8} & -\frac{1}{8} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & 0 & 0 \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{6} & \frac{1}{6} & 0 & 0 \end{pmatrix}$$
$$= (1, 0, 1, 0, 0, 1). \tag{72}$$

The resulting row vector indicates that Eq. (63) for this case is represented as follows:

$$\mathbf{T}_h(/\mathbf{C}_{2v}) = A_g + E_g + T_u. \tag{73}$$

Hence, we use the A_g, E_g, and T_u rows of Table 9, where the characteristic monomials of each subsymmetry are multiplied to give an SCI in accord with Eq. (64). For example, we have $s_1 \times s_1^{-1}s_3 \times s_3^{-1}s_6 = s_6$, for the $\mathbf{S}_6$ column. Thereby, we obtain the corresponding CI,

$$\mathrm{CI}(\mathbf{T}_h; s_d) = \tfrac{1}{24}s_1^6 + \tfrac{1}{8}s_1^2 s_2^2 + \tfrac{1}{8}s_1^4 s_2 + \tfrac{1}{24}s_2^3 + \tfrac{1}{3}s_3^2 + \tfrac{1}{3}s_6, \tag{74}$$

by means of Eq. (65). Suppose that the variable x represents unchanged cyclopropanes while the variable y represents changed cyclopropanes. Then, we have an inventory,

$$s_d = x^d + y^d, \tag{75}$$

which is introduced into Eq. (74). The resulting equation is expanded to give the following generating function:

$$f = (x^6 + y^6) + (x^5 y + xy^5) + 2(x^4 y^2 + x^2 y^4) + 3x^3 y^3, \tag{76}$$

where each coefficient of the term $x^m y^n$ indicates the number of isomers having m unchanged and n changed cyclopropane rings ($m + n = 6$). To illustrate the result, the derivatives for $n = 1$–3 are depicted in Fig. 1.

It should be noted that if a derivative is chiral an arbitrary enantiomer is depicted in Fig. 1.

5 Characteristic monomial tables to (non)dominant USCI tables

In a previous paper [30], we defined USCI tables for dominant and nondominant representations in terms of the subduction of dominant and nondominant representations. In the light of the present method, such USCI tables can be alternatively obtained by virtue of the reduction of dominant and nondominant representations. Thus, the permutation $\mathbf{P}$ in Eq. (63) is replaced by a dominant or nondominant representation represented by $\mathbf{G}(/\mathbf{G}_j)$. Thereby, we have

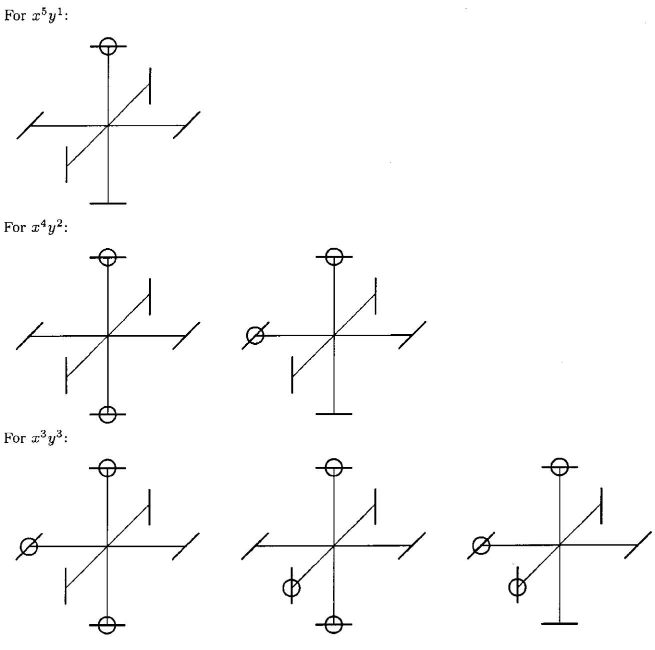

Fig. 1. Derivatives form the skeleton 1

$$\mathbf{G}(/\mathbf{G}_j) = \sum_i \beta_i \Gamma_i. \tag{77}$$

In a similar way as described for Eq. (64), the corresponding USCI can be calculated by

$$\mathrm{USCI}(\mathbf{G}(/\mathbf{G}_j) \downarrow \mathbf{C}_u; s'_d) = \prod_i \left[Z(\Gamma_i \downarrow \mathbf{C}_u; s'_d)\right]^{\beta_i}. \tag{78}$$

The resulting USCIs are applicable to the enumeration described in Ref. [30].

Example 9. This is a continuation of Example 8 concerning the $\mathbf{T}_h$ group. In a similar way as described for deriving Eqs. (72) and (73), each dominant or non-dominant representation is reduced into a sum of $\mathbf{Q}$-conjugacy representations. Thus, for dominant representations, we have the following results:

$$\mathbf{T}_h(/\mathbf{C}_1) = A_g + A_u + E_g + E_u + 3T_g + 3T_u,$$
$$\mathbf{T}_h(/\mathbf{C}_2) = A_g + A_u + E_g + E_u + T_g + T_u,$$
$$\mathbf{T}_h(/\mathbf{C}_s) = A_g + E_g + T_g + 2T_u,$$
$$\mathbf{T}_h(/\mathbf{C}_i) = A_g + E_g + 3T_g,$$
$$\mathbf{T}_h(/\mathbf{C}_3) = A_g + A_u + T_g + T_u,$$
$$\mathbf{T}_h(/\mathbf{S}_6) = A_g + T_g.$$

On the other hand, we have the following reductions for nondominant representations:

$$\mathbf{T}_h(/\mathbf{C}_{2v}) = A_g + E_g + T_u,$$
$$\mathbf{T}_h(/\mathbf{C}_{2h}) = A_g + E_g + T_g,$$
$$\mathbf{T}_h(/\mathbf{D}_2) = A_g + A_u + E_g + E_u,$$
$$\mathbf{T}_h(/\mathbf{D}_{2h}) = A_g + E_g,$$
$$\mathbf{T}_h(/\mathbf{T}) = A_g + A_u,$$
$$\mathbf{T}_h(/\mathbf{T}_h) = A_g.$$

Table 11. Unit subduced cycle indices (*USCIs*)

$\mathbf{T}_h$	$\downarrow \mathbf{C}_1$	$\downarrow \mathbf{C}_2$	$\downarrow \mathbf{C}_s$	$\downarrow \mathbf{C}_i$	$\downarrow \mathbf{C}_3$	$\downarrow \mathbf{S}_6$
(Dominant USCIs)						
$\mathbf{T}_h$ (/$\mathbf{T}$)	s_1^2	s_1^2	s_2	s_2	s_1^2	s_2
$\mathbf{T}_h$ (/$\mathbf{C}_1$)	s_1^{24}	s_1^{12}	s_1^{12}	s_1^{12}	s_1^8	s_1^4
$\mathbf{T}_h$ (/$\mathbf{C}_2$)	s_1^{12}	$s_1^4 s_2^4$	s_2^6	s_2^6	s_3^4	s_6^2
$\mathbf{T}_h$ (/$\mathbf{C}_s$)	s_1^{12}	s_2^6	$s_1^4 s_2^4$	s_2^6	s_3^4	s_6^2
$\mathbf{T}_h$ (/$\mathbf{C}_i$)	s_1^{12}	s_2^6	s_2^6	s_1^{12}	s_3^4	s_3^4
$\mathbf{T}_h$ (/$\mathbf{C}_3$)	s_1^8	s_2^4	s_2^4	s_2^4	$s_1^2 s_3^2$	$s_2 s_6$
$\mathbf{T}_h$ (/$\mathbf{S}_6$)	s_1^4	s_2^2	s_2^2	s_1^4	$s_1 s_3$	$s_1 s_3$
(Non dominant USCIs)						
$\mathbf{T}_h$ (/$\mathbf{C}_{2v}$)	s_1^6	$s_1^2 s_2^2$	$s_1^4 s_2$	s_2^3	s_3^2	s_6
$\mathbf{T}_h$ (/$\mathbf{C}_{2h}$)	s_1^6	$s_1^2 s_2^2$	$s_1^2 s_2^2$	s_1^6	s_3^2	s_3^2
$\mathbf{T}_h$ (/$\mathbf{D}_2$)	s_1^6	s_1^6	s_2^3	s_2^3	s_3^2	s_6
$\mathbf{T}_h$ (/$\mathbf{D}_{2h}$)	s_1^3	s_1^3	s_1^3	s_1^3	s_3	s_3
$\mathbf{T}_h$ (/$\mathbf{T}$)	s_1^2	s_1^2	s_2	s_2	s_1^2	s_2
$\mathbf{T}_h$ (/$\mathbf{T}_h$)	s_1	s_1	s_1	s_1	s_1	s_1
N_u	$\frac{1}{24}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{3}$	$\frac{1}{3}$

In the same way as shown in Example 9 for the representation $\mathbf{T}_h(/\mathbf{C}_{2v})$ where Eq. (73) gives respective monomials appearing in the CI (Eq. 73), each of the reduction data listed above give USCIs for a dominant or nondominant representation, as collected in Table 11. The USCIs in Table 11 are identical with the counterparts selected from the full USCI table (Table 6 in Ref. [37]).

6 Conclusion

We have discussed

1. Amplificative equivalence of coset representations and irreducible representations.
2. Number-theoretical vectors.
3. The Möbius function.

The first and second concepts have clarified the interconversion between dominant markaracters and **Q**-conjugacy characters for cyclic groups. The third one has given characteristic monomials for a cyclic group, which are in turn used to build up characteristic monomials for a finite group. This process indicates that all the powers appearing in a characteristic monomial are integers. The characteristic monomials are applied to the enumeration of isomers.

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References

1. Cotton FA (1971) Chemical applications of group theory. Wiley, New York
2. Jaffé HH, Orchin M (1965) Symmetry in chemistry. Wiley, Chichester
3. Hall LH (1969) Group theory and symmetry in chemistry. McGraw-Hill, New York
4. Bishop DM (1973) Group theory and chemistry. Clarendon Press, Oxford
5. Kettle SFA (1985) Symmetry and structure. Wiley, Chichester
6. Ladd MFC (1989) Symmetry in molecules and crystals. Ellis Horwood, Chichester
7. Harris DC, Bertolucci MD (1989) Symmetry and spectroscopy. Dover, New York
8. Hargittai I, Hargittai H (1986) Symmetry through the eyes of a chemist. VCH, Weinheim
9. Tsukerblat BS (1994) Group theory in chemistry and spectroscopy. Academic Press, New York
10. Pólya G (1937) Acta Math 68: 145–254
11. Harary F, Palmer EM (1973) Graphical enumeration. Academic Press, New York
12. Balaban AT (ed) (1976) Chemical applications of graph theory. Academic Press, London
13. Davidson RA (1981) J Am Chem Soc 103: 312–314
14. Pólya G, Read RC (1987) Combinatorial enumeration of groups, graphs, and chemical compounds. Springer Berlin Heidelberg New York
15. Balasubramanian K (1985) Chem Rev 85: 599–618
16. Hässelbarth W (1985) Theor Chim Acta 67: 339–367
17. Brocas J (1986) J Am Chem Soc 108: 1135–1145
18. Mead CA (1987) J Am Chem Soc 109: 2130–2137
19. Lloyd EK (1992) J Math Chem 11: 207–222
20. Burnside W (1911) Theory of groups of finite order, 2nd edn. Cambridge University Press, Cambridge
21. Sheehan J (1968) Can J Math 20: 1068–1076
22. Kerber A, Thürlings KJ (1982) In: Jugnickel D, Vedder K (eds) Combinatorial theory. Springer Berlin Heidelberg New York pp 191–211
23. Redfield JH (1984) J Graph Theory 8: 205–223
24. Fujita S (1989) Theor Chim Acta 76: 247–268
25. Fujita S (1991) Symmetry and combinatorial enumeration in chemistry. Springer Berlin Heidelberg New York
26. Fujita S (1992) Theor Chim Acta 82: 473–498
27. Fujita S (1993) J Math Chem 12: 173–195
28. Fujita S (1994) J Graph Theory 18: 349–371
29. Fujita S (1995) Theor Chim Acta 91: 291–314
30. Fujita S (1995) Theor Chim Acta 91: 315–332
31. Fujita S (1998) Theor Chem Acc 99: 224–230
32. (a) Fujita S (1998) Bull Chem Soc Jpn 71: 1587–1596; (b) Fujita S (1998) Bull Chem Soc Jpn 71: 2071–2080; (c) Fujita S (1998) Bull Chem Soc Jpn 71: 2309–2321
33. Takagi T (1971) Shoto seisu-ron kogi, 2nd edn. Sankyo, Tokyo
34. Serre JP (1977) Linear representations fo finite groups. Springer Berlin Heidelberg New York
35. (a) Lovász L (1979) Combinatorial problems and exercises. Akadémiai Kaidó, Budapest; (b) Berge C (1971) Principles of combinatorics. Academic Press, London; (c) Aiger M (1979) combinatorial theory. Springer Berlin Heidelberg New York
36. Fujita S (1990) J Math Chem 5: 99–120
37. Fujita S (1996) Chem Inf Comput Sci 36: 270–285